

Resolution for Quantified Signed Formulae

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[Abstract] This paper is devoted to investigate the resolution for quantified signed formulae (signed formulae with quantifiers). First we introduce quantified signed formulae. Then we investigate resolution for quantified signed formulae and prove its soundness and refutational completeness. Also a restricted resolution (binary resolution) is introduced and proved to be sound and refutationally complete for a subclass of quantified signed formulae which each formula is with regular signed formulae as matrices (regular quantified signed formulae). As a result, the subclass of regular quantified signed formulae, which each formula only contains clauses with at most two signed literals, is tractable.

[Key words] resolution; quantified signed formulae; regular; tractable

量化带标公式的消解

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[摘要] 首先引入量化带标公式, 然后研究了量化带标公式的消解并且证明其健全性和拒绝完备性. 另外, 还引入了二元消解并证明其针对正规量化带标公式(一个量化带标公式的子集)是健全的和拒绝完备的. 最后证明如果正规量化带标公式的每一个子句如果最多包含两个文字, 则该公式的可满足性问题是易解的.

[关键词] 消解; 量化带标公式; 正规; 易解

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1 Introduction

Recently besides the satisfiability problem of boolean formulae (SAT) and constraintsatisfaction problems (CSP), the satisfiability problem of signed formulae have attracted a lot of interests. Signed formulae are classical conjunctive clause forms using signed literals. A signed literal is an expression of the form $S:v$ where v is a variable and S is a subset of the domain D_v of v . The informal meaning of $S:v$ is that v takes one of the values in S . Signed formulae expand the ability of

expression of boolean formulae, and have been studied recently and yielded some interesting results (e. g., [1-6]).

Parallel to quantified boolean CNF formulae (QCNF) and quantified constraint satisfaction problems (QCSP), it is natural to introduce quantified signed formulae, which can be consider as the generalization of signed formulae. Quantified signed formulae can be used to express some AI problems in a natural way. For example some game problems, it is natural to use \forall and \exists to express the players and signed formulae to

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express the game rules. We will give an example in next section. For signed formulae, transferring them to boolean CNF formulae is a useful method to solve the satisfiability problem of them (e. g., [2]). Unfortunately this method seems hopeless to be used for quantified signed formulae because of quantifiers. This motivate us to develop new method to solve the satisfiability problem of quantified signed formulae.

There have been some important algorithms which are proved to be powerful to solve some hard instances of satisfiability problem. Among them, resolution approach plays an important role. Resolution for boolean CNF formulae was firstly introduced in [7], and has been studied widely from then on. The generalization of resolution to quantified boolean CNF formulae, called Q-resolution, was first introduced in [8]. Resolution and Q-resolution have been not only applied in some SAT or QSAT solvers (e. g., [9–10]), but also adopted to prove some theoretical results (e. g., [11–13]). The generalization of resolution to signed formulae has been studied recently (e. g., [3,5]).

Just as Q-resolution for QCNF, we generalize resolution for signed formula to general quantified resolution (G-Q-resolution, for short) for quantified signed formulae. G-Q-resolution can be used to refute false quantified signed formulae. And if the quantified signed formula has some special matrix structure, some restricted G-Q-resolution can refute it in polynomial time.

This paper is organized as follows. In next section we introduce quantified signed formulae and some terminologies used in this paper. Section 3 introduces G-Q-resolution for quantified signed formulae and prove that G-Q-resolution is sound and refutationally complete. Section 4 introduces a subclass of quantified signed formulae (regular quantified signed formulae). Regular quantified signed formulae are quantified signed formulae with regular signed formulae as matrices. Also in this section a restricted G-Q-resolution, binary general quantified resolution (B-G-Q-resolution for short), is introduced. B-G-Q-resolution is sound and refutationally complete for regular quantified signed formulae. As a result if each clause in a regular quantified signed formulae contains at most two signed liter-

als, this formulae can be decided the satisfiability in polynomial time. Section 5 concludes this paper and outlines the future work.

2 Quantified Signed Formulae

First we recall the definition of signed formulae and some terminologies which will be used later on.

The universe of variables is a countably infinite set denoted by VA , and the universe of domain is denoted by DOM . Each variable $v \in VA$ has a non-empty (value-) domain $D_v \subseteq DOM$.

For each $v \in VA$ and each subset $S \subseteq D_v$, $S : v$ is termed as a (general) literal, S is the literal's sign. Usually, we use x, y, z (or with subscripts) to denote literals. The informal meaning of literal $S : v$ is that v takes one value in S . A (general) clause is the disjunction of literals, we also consider it as the set of literals. A signed formula is the conjunction of clauses (also be considered as the set of clauses).

A (partial) assignment φ is a map from some subset of VA to DOM such that for every variable $v \in dom(\varphi)$ we have $\varphi(v) \in D_v$. Given an assignment φ , we have:

1. For a literal $S : v$, if $v \in dom(\varphi)$ and $\varphi(v) \in S$ (resp. $\varphi(v)$ not in S) we say that $S : v$ is true or $S : v = 1$ (resp. $S : v$ is false or $S : v = 0$) under φ .
2. For a clause C , if there is a literal $S : v$ in C such that $S : v$ is true, we say C is true under φ . If all literals in C are false under φ , we say C is false under φ .
3. For a signed formula F , if all it's clauses are true, we say F is true under φ . If there is one clause in F is false, we say F is false under φ .

For a signed formula F , if there exists a assignment makes F true, we say F is satisfiable, otherwise we say F is unsatisfiable.

Assignments also can be used to reduce a formulae. For an assignment φ and a signed formula F , we get the reduced formula $\varphi * F$ by: first delete all clauses which are true under φ , then delete all literals which are false under φ from remaining clauses. Obviously if F is true under φ then $\varphi * F$ is the empty formula (denoted by \top), if F is false then $\varphi * F$ contains the empty clause (denoted by \perp). For conven-

ience we use $F_{v=\varepsilon}$ to represent the reduced formula $\varphi * F$ when $\text{dom}(\varphi) = \{v\}$ and $\varphi(v) = \varepsilon$.

Then we give the definition of quantified signed formulae, in generally quantified signed formulae are formulae with quantifiers.

Definition 1 The class of quantified signed formulae is the least set satisfying the following conditions:

(1) Every signed formulae is a quantified signed formula.

(2) For a quantified signed formulae Φ and a variable v occurring in Φ , both $\forall v\Phi$ and $\exists v\Phi$ are quantified signed formula.

It is easy to see that quantified signed formulae have structures like $Q_1v_1 \cdots Q_nv_n \alpha$, here $Q_i \in \{\exists, \forall\}$ and v_i is a variable for $i \in \{1, \dots, n\}$, α is a signed formula. We use upper case Greek letters such as Φ , Ψ (or with subscripts) to denote quantified signed formula. The $Q_1v_1 \cdots Q_nv_n$ part is called the prefix, α is the matrix. Sometimes we use an abbreviation and write $\Phi = Q\alpha$. The variables following \exists (resp. \forall) directly are called existential (resp. universal) variables, literals on existential (resp. universal) variables are called existential (resp. universal) literals. For a quantified signed formula $\Phi = Q_1v_1 \cdots Q_nv_n \alpha$, we use $\text{var}(\Phi)$ to denote the set of variables occurring in Φ . By the definition we know that for a quantified signed formula all variables occurring in prefix must occur in matrix.

For a quantified signed formula $\Phi = Q_1v_1 \cdots Q_nv_n \alpha$, if all variables occurring in matrix also occur in prefix, we say that Φ is closed. If Φ is not closed, the variable occurring in matrix but not occurring in prefix is called free variable. The set of all free variables in Φ is denoted by $\text{free}(\Phi)$. Whereas variables occurring the prefix are called bounded variables. For our later purpose, we define an (partial) order " $<$ " on variables of Φ . For two bounded variables v_i, v_j , we say v_i is smaller than v_j if $i < j$, that is, v_i is on the left of v_j in the prefix; free variables are always smaller than bounded variables. This order can be extended to literals as follows: a literal x is smaller than a literal y if $\text{var}(x)$ is smaller than $\text{var}(y)$.

Next we give an instance of a two-person coloring game, which expand a instance in [14] from two col-

ors to n colors. We will find that using quantified signed formulae to code such problems is more naturally.

Example 1 Given a set of variables $A = \{v_1, \dots, v_{2k}\}$ with order $v_i < v_j$ for $i < j$. And a collection $C = \{c_1, \dots, c_m\}$ of subset of A . Two players, \forall and \exists , color the variables with n colors $CL = \{cl_1, \dots, cl_n\}$ by the order of A in turn, and player \forall first. Player \exists win if and only if when all variables in A are colored, each element of C don't with all variables in same color.

This problem can be coded to a quantified signed formulae $Q\alpha$. Here Q is the prefix $\forall v_1 \exists v_2 \cdots \forall v_{2k-1} \exists v_{2k}$. And $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_m$ with each α_i represents that c_i don't with all its variables in same color, for example, if $c_i = \{v_1, v_2\}$ then $\alpha_i = (CL \setminus \{cl_1\} : v_1 \vee CL \setminus \{cl_1\} : v_2) \wedge \cdots \wedge (CL \setminus \{cl_n\} : v_1 \vee CL \setminus \{cl_n\} : v_2)$, or in set form $\{\{CL \setminus \{cl_1\} : v_1, CL \setminus \{cl_1\} : v_2\}, \dots, \{CL \setminus \{cl_n\} : v_1, CL \setminus \{cl_n\} : v_2\}\}$. Player \exists win if and only if $Q\alpha$ is satisfiable.

For a quantified signed formula Φ and an assignment φ . We define the value of Φ as follows.

Definition 2 Suppose Φ is a quantified signed formula, φ is an assignment such that $\text{free}(\Phi) \subseteq \text{dom}(\varphi)$. The value of Φ under φ , denoted by $V_\varphi(\Phi)$, is defined inductively as follows:

1. If the prefix of Φ is empty, i. e., Φ is a signed formula α , then

When $\varphi * \alpha = \top$, $V_\varphi(\Phi) = 1$;

When \perp is contained in $\Phi * \alpha$, $V_\varphi(\Phi) = 0$.

2. If $\Phi = \exists vQ\alpha$, then $V_\varphi(\Phi) = \max \{V_\varphi(Q\alpha_{v=\varepsilon}) : \varepsilon \in D_v\}$.

3. If $\Phi = \forall vQ\alpha$, then $V_\varphi(\Phi) = \min \{V_\varphi(Q\alpha_{v=\varepsilon}) : \varepsilon \in D_v\}$.

We say Φ is satisfiable if there is some assignment φ such that $V_\varphi(\Phi) = 1$. Otherwise Φ is unsatisfiable.

From Definition 2 we can see that the values of bounded variables under an assignment have no effect on the value of the quantified signed formula. Especially, if Φ is a closed quantified signed formula, then Φ has the same value under all assignments, either 1 (true) or 0 (false). Thus, for closed quantified signed formulae we call them true or false instead of satisfiable or unsatisfiable.

For a variable $v \in \text{var}(\Phi)$, if D_v contains exactly

one value ε , then any assignment with v in its domain must assign ε to v , and hence any occurrence of the literal $S : v$ can be removed without changing the satisfiability. So, in this paper, we demand $|D_v| \geq 2$ for any variable v . If a literal $S : v$ with $S = D_v$, clauses with such literal (we call them full domain literals) are tautology and can be removed quantified signed formulae without changing satisfiability, so we demand all quantified signed formulae mentioned later contain no full domain literals.

Suppose Φ is a quantified signed formula with free variables among $\{v_1, \dots, v_k\}$. From Definition 2 it is easy to see the following proposition.

Proposition 1 Φ is satisfiable if and only if $\exists v_1 \dots \exists v_k \Phi$ is true.

Since each variable can only take a fixed number of values, it is easy to see that the problem of determining the satisfiability of quantified signed formulae is in PSPACE. On the other hand, quantified boolean CNF formulae are also quantified signed formulae, so we get the following proposition.

Proposition 2 The satisfiability problem for quantified signed formulae is PSPACE-complete.

Definition 3 Two quantified signed formulae Φ and Ψ are equivalent, in symbols $\Phi \approx \Psi$, if and only if for every assignment φ such that $\text{free}(\Phi) \cup \text{free}(\Psi) \subseteq \text{dom}(\varphi)$ we have $V_\varphi(\Phi) = V_\varphi(\Psi)$.

It is easy to see that equivalent transformations of the matrix do not affect the satisfiability of the whole quantified signed formula.

Proposition 3 Let Q be a prefix, Φ and Ψ quantified signed formulae. Then $\Phi \approx \Psi \Rightarrow Q\Phi \approx Q\Psi$

Let $\Phi = Q\alpha$ be a quantified signed formula, and $C = \{x_1, \dots, x_k\}$ is a clause in α such that $x_1 < \dots < x_k$. Please note that if x_k is a universal literal then deleting x_k from C will not change the value of Φ . Then the following proposition is obvious.

Proposition 4 Suppose a quantified signed formula $\Phi = Q\alpha$ and C_1, \dots, C_n are some clauses in α . For each $i = 1, \dots, n$, we remove all universal literals in C_i which are not smaller than any existential literal in C_i , the resulting clauses are denoted by C'_i . We have

$$\Phi \approx Q((\alpha \setminus \{C_1, \dots, C_n\}) \cup \{C'_1, \dots, C'_n\}).$$

3 Resolution for Quantified Signed Formulae

Firstly we recall the resolution for signed formulae in [15].

For two clauses $C_1 = \{S_1 : v\} \cup D_1$ and $C_2 = \{S_2 : v\} \cup D_2$, we can get a resolvent $\{S_1 \cap S_2 : v\} \cup D_1 \cup D_2$. This operation is called a resolution step. Of course if there are some literals $S'_1 : v', \dots, S'_m : v'$ on a same variable v' in the resolvent, we can disjunct signs S'_1, \dots, S'_m and get a new literal $S'_1 \cup \dots \cup S'_m : v'$. We call this operation literals unite step. Literals unite step can yield some full domain literals.

It have been proved that the resolution above is a refutation complete calculus for signed formulae [15]. Unlike the resolution for boolean CNF formulae, the literals resolved upon doesn't necessarily vanish.

Next we generalize the resolution for signed formula to G-Q-resolution for quantified signed formulae and prove it is sound and refutationally complete. Our generalization is a little differ from [15] in: parents clauses can be more than two, and the literals resolved upon must vanish.

Definition 4 For a quantified signed formula $\Phi = Q\alpha$, if there is a existential variable v and a group of clauses $C_1 = D_1 \cup \{S_1 : v\}, \dots, C_n = D_n \cup \{S_n : v\}$ in α such that $S_1 \cap \dots \cap S_n = \emptyset$. We define a clause R as follows.

1. From each $C_i (i \in \{1, \dots, n\})$ delete all universal literals which are not smaller than any existential literal in C_i . These new clauses are denoted by $D'_1 \cup \{S_1 : v\}, \dots, D'_n \cup \{S_n : v\}$.
2. Construct a new set $R := D'_1 \cup \dots \cup D'_n$ and do literals unite steps.

If the new set R contains no full literals, the operation above is called a G-Q-resolution step, R is called the resolvent, and C_1, \dots, C_n are called the parent clauses. In symbols, $\Phi \vdash_{G-Q}^1 R$.

By Proposition 3 and Proposition 4, the soundness, that is, adding the resolvent to a quantified signed formula does not affect the truth, is easy to prove.

Theorem 1 For a quantified signed formula $\Phi = Q\alpha$ and a clause R , if $\Phi \vdash_{G-Q}^1 R$, then $Q\alpha \approx Q(\alpha \cup$

$\{R\}$).

A clause without existential literals is called a universal clause. Then the empty clause \perp is also a universal clause. Obviously, if a quantified signed formula contains a universal clause without full literals, it is always false.

We conclude this section by proving the refutational completeness of G-Q-resolution for quantified signed formulae. Please remind that we have supposed there are not full domain literals in the (original) quantified signed formula. Also we suppose the (original) quantified signed formula is closed, by Proposition 1 we can see this is not a restriction.

Theorem 2 *For any quantified signed formula Φ , Φ is false if and only if some universal clause R without full domain literals can be obtained by iterating G-Q-resolution steps.*

Proof. The direction from right to left follows from Theorem 1. Now we prove the inverse direction by induction on the length of the prefix.

Suppose $\Phi = \exists v\alpha$ with α a signed formula. Since Φ is false, α is unsatisfiable. So, α must contain the formula $\{\{S_1:v\}, \dots, \{S_n:v\}\}$ with $S_1 \cap \dots \cap S_n = \emptyset$. It is easy to see that $\Phi \vdash_{G-Q}^1 \perp$.

Suppose $\Phi = \forall v\alpha$ with α a signed formula. Then all clauses in α are universal clauses. Since Φ is false, α is non-empty. So, there is a $C \in \alpha$, and we have $\Phi \vdash_{G-Q}^1 C$.

Suppose $\Phi = \forall vQ\alpha$. Because Φ is false, so for each $\varepsilon_i \in D_v = \{\varepsilon_1, \dots, \varepsilon_n\}$, the formula $Q(\alpha_{v=\varepsilon_i})$, denoted by Φ_{ε_i} , is false. We remind that Φ_{ε_i} is obtained from $Q\alpha$ by (1) deleting all clauses containing a literal $S:v$ with such that $\varepsilon_i \in S$, and (2) deleting all occurrences of the literals $S:v$ such that ε_i is not in S from the remaining clauses. By the induction hypothesis we can get a universal clause R_{ε_i} from Φ_{ε_i} by G-Q-resolution. By recovering the occurrences of the literal lost, we have the resolvent either R_{ε_i} or $\{S_i:v\} \cup R_{\varepsilon_i}$ for some S_i such that ε_i is not in S_i . If the first case holds, then the theorem follows. So, we assume that we get $\{S_i:v\} \cup R_{\varepsilon_i}$ for each $\varepsilon_i \in D_v$. Then we resolve clauses $\{S_i:v\} \cup R_{\varepsilon_1}, \dots, \{S_n:v\} \cup R_{\varepsilon_n}$.

Because ε_i is not in S_i for $i \in \{1, \dots, n\}$, so $S_1 \cap \dots \cap S_n = \emptyset$. Then we obtain the empty clause \perp .

Suppose $\Phi = \forall vQ\alpha$. Because Φ is false, there must be a $\varepsilon_i \in D_v$ such that the formula Φ_{ε_i} is false, here Φ_{ε_i} is obtained by the same way as in the above paragraph. By the induction hypothesis we get some universal clause R_{ε_i} . By the same argument as in the above paragraph, we have either R_{ε_i} or $\{S_i:v\} \cup R_{\varepsilon_i}$. Both resolvents are universal clauses without full domain literals, and we complete the proof. \square

4 Resolution for Regular Quantified Signed Formulae

Regular signed formulae are signed formulae with regular sign and have been investigated recently (e.g., [1,2,6]). This section is focus on resolution for regular quantified signed formulae. First let us recall the definition of regular signed formulae.

Regular signed formulae are signed formulae with the following characteristic: for each variable v , D_v is equipped with a total order \leq , for each literal $S:v$ (regular literal), S (regular sign) must be either $\{j \in D_v \mid j \leq i\}$ or $\{j \in D_v \mid j \geq i\}$ for some $i \in D_v$. For convenience we use $\uparrow i$ (resp. $\downarrow i$) to represent $\{j \in D_v \mid j \geq i\}$ (resp. $\{j \in D_v \mid j < i\}$). In addition, clauses with only regular literals are called regular clauses.

In order to refute unsatisfiable regular signed formulae, a restricted resolution, named binary resolution, is introduced [1]. A binary resolution step is such a operation: for two clauses $C_1 = D_1 \cup \{\uparrow i:v\}$ and $C_2 = D_2 \cup \{\downarrow j:v\}$, if $i \geq j$ then we get the resolvent $D_1 \cup D_2$. It has been proved that binary resolution is sound and refutationally complete for regular signed formulae [1].

Note: Indeed [1] have proved a more strong result, that is binary resolution is sound and refutationally complete for many-valued CNF formulae. Regular signed formulae is a subset of many-valued CNF formulae.

Next we introduce regular quantified signed formulae and binary general quantified resolution (B-G-Q-resolution for short). Then prove that B-G-Q-resolution is sound and refutationally complete for regular quantified signed formulae.

Definition 5 *A regular quantified signed formula is a quantified signed formula with regular signed formula as matrix.*

From the definition above, regular quantified signed formulae is a subset of quantified signed formulae, so theorems and propositions for quantified signed formulae in former sections still hold for regular quantified signed formulae.

Unlike quantified signed formulae, we consider regular quantified signed formulae with free variables in the rest of this paper, because this can make our proof more clearly.

B-G-Q-resolution is similar to G-Q-resolution, the definition is as follows.

Definition 6 For a regular quantified signed formula $\Phi = Q\alpha$, if there exists a existential variable v and two clauses $C_1 = D_1 \cup \{\uparrow i:v\}$ and $C_2 = D_2 \cup \{\downarrow j:v\}$ in α such that $i \geq j$, we get a clause R as follows.

1. From C_1 and C_2 delete all universal literals which are not smaller than any existential literal. These new clauses are denoted by $D'_1 \cup \{\uparrow i:v\}$ and $D'_2 \cup \{\downarrow j:v\}$.

2. Construct a new set $R := D'_1 \cup D'_2$.

If the new set R don't contain any literals $\uparrow i:v'$ and $\downarrow j:v'$ such that $i \leq j$, then the operation above is called a B-G-Q-resolution step, R is called the resolvent, and C_1 and C_2 are called the parent clauses. In symbols, $\Phi \vdash_{B-G-Q-res}^1 R$.

Note: In order to make the resolvent still regular, unlike G-Q-resolution, we don't unite literals in B-G-Q-resolution steps. But if a resolvent contains two literals $\uparrow i:v$ and $\downarrow j:v$ such that $i \leq j$, this resolvent is useless for refute an unsatisfiable formula. So we discard such resolvents.

Although we don't unite literals as G-Q-resolution steps, another uniting is necessary. If a resolvent contains two literals $\uparrow i:v$ and $\uparrow j:v$ ($\downarrow i:v$ and $\downarrow j:v$) such that $i \leq j$ ($i \geq j$), we can unite the two literals to one literal $\uparrow i:v$ ($\downarrow i:v$), this operation is called polar literals unite step.

Next we will prove that B-G-Q-resolution is sound and refutationally complete for regular quantified signed formulae.

Theorem 3 For any regular quantified signed formula Φ , Φ is false if and only if some universal clause R can be obtained by iterating B-G-Q-resolution steps.

Proof. The direction from right to left is obvious, just as Theorem 2. Now we prove the inverse direction by

induction on the length of the prefix.

Suppose the prefix of Φ is empty, that is $\Phi = \alpha$ with α a regular signed formula. It have been prove that α can be refuted by binary resolution [1]. In such situation (the prefix is empty), binary resolution is also B-G-Q-resolution.

Suppose $\Phi = \exists v Q\alpha$. Because Φ is false, so for each $\varepsilon_i \in D_v = \{\varepsilon_1, \dots, \varepsilon_n\}$, the formula $Q(\alpha_{\varepsilon_i})$, denoted by Φ_{ε_i} , is false. Just as in Theorem 2, Φ_{ε_i} is obtained from $Q\alpha$ by (1) deleting all clauses containing a literal $S:v$ with such that $\varepsilon_i \in S$, here $S = \uparrow i$ or $\downarrow i$ for some i , and (2) deleting all occurrences of the literals $S:v$ such that ε_i is not in S from the remaining clauses. By the induction hypothesis we can get a universal clause R_{ε_i} from Φ_{ε_i} by B-G-Q-resolution. By recovering the occurrences of the literal lost, we have the resolvent either R_{ε_i} or $\{\uparrow j_i:v, \downarrow k_i:v\} \cup R_{\varepsilon_i}$ for some j_i and k_i (Maybe only j_i or k_i). If the first case holds, then the theorem follows. So, we assume that we get $\{\uparrow j_i:v, \downarrow k_i:v\} \cup R_{\varepsilon_i}$ for each $\varepsilon_i \in D_v$. Because ε_i is not in $\uparrow j_i \cup \downarrow k_i$ for $i \in \{1, \dots, n\}$, so $\{\uparrow j_1:v, \downarrow k_1:v\}, \dots, \{\uparrow j_n:v, \downarrow k_n:v\}$ can be considered as a unsatisfiable regular signed formula. So from the definition of B-G-Q-resolution and the case that unsatisfiable regular signed formula can be refuted by binary resolution, we can get a B-G-Q-resolution step such that $\{\{\uparrow j_1:v, \downarrow k_1:v\} \cup R_{\varepsilon_1}, \dots, \{\uparrow j_n:v, \downarrow k_n:v\} \cup R_{\varepsilon_n}\} \vdash_{B-G-Q-res}^1 \perp$. (Please note: a B-G-Q-resolution step is same as a binary resolution step after deleting universal literals which are not smaller than any existential literal)

Suppose $\Phi = \forall v Q\alpha$. Because Φ is false, there must be a $\varepsilon_i \in D_v$ such that the formula Φ_{ε_i} is false, here Φ_{ε_i} is obtained by the same way as in the above paragraph. By the induction hypothesis we get some universal clause R_{ε_i} . By the same argument as in the above paragraph, we have either R_{ε_i} or $\{\uparrow j_i:v, \downarrow k_i:v\} \cup R_{\varepsilon_i}$. Both resolvents are universal clauses, and we complete the proof. \square

For a unsatisfiable regular signed formula which each clause contains at most two literals, binary resolution can refute it in polynomial time. This is because that when applying binary resolution to a regular signed formula, the resolvent contains at most two literals, so

the number of resolvents can be bounded by polynomial. In the same way, B-G-Q-resolution can refute regular quantified signed formulae which each clause contains at most two literals.

Let us end this section by the obvious theorem.

Theorem 4 *For a regular quantified signed formula Φ , if each clause of it has at most two literals, then the satisfiability of Φ can be decided in polynomial time by B-G-Q-resolution.*

5 Conclusion and Future Work

In this paper we introduce quantified signed formulae and investigate resolution for quantified signed formulae, also we introduce a subset of quantified signed formulae which can be decided in polynomial time. Many problems can be coded to quantified signed formulae. Even some can be coded to regular quantified signed formulae which each clause contains at most two literals, then they can be decided the satisfiability in polynomial time.

We believe that quantified signed formulae and regular quantified signed formulae will play an important role in AI in future, so our future work is to find some new method to solve the satisfiability problem of quantified signed formulae and regular quantified signed formulae. Also we will try to find some other tractable subclass of quantified signed formulae.

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